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The trapping of surface waves above a submerged, horizontal cylinder

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The existence of surface waves trapped above a submerged horizontal cylinder was shown by Ursell to depend upon the vanishing of a certain infinite determinant. Here, the determinant is evaluated numerically and the dispersion curves found. It is shown that the mild slope equation may be used to determine the dispersion relation and surface profiles with good accuracy and with less computational effort than the full linear theory.

1. Introduction

A limited number of explicit solutions are known for water waves trapped by the bottom topography in such a way that the amplitude of the motion decays to zero at large distances. The first such edge wave was discovered by Stokes (1846) who showed that a wave may progress along a straight coastline with the motion decaying exponentially in the offshore direction in the case where the offshore bed profile is a plane beach of constant slope. This work was extended by Ursell (1952) who gave explicit solutions for a set of edge-wave modes on a plane beach of which the fundamental mode is that found by Stokes. The *existence* of a trapping mode above a submerged, horizontal, circular cylinder was first proved by Ursell (1951). On the basis of the full linearized theory of water waves, he showed that the existence of trapped waves depended upon the vanishing of a certain infinite determinant. He went on to show that zeros of the determinant exist if the radius of the cylinder was small compared to the length of the waves. This is not a physical restriction, as has been shown by Jones (1953), who proved that trapped waves exist for a number of geometries including a submerged cylinder of any radius and a rectangular shelf adjoining a region of greater depth. The full linear solution for the shelf has been determined semi-analytically by Evans & McIver (1984).

Approximate solutions based upon shallow-water theory have been obtained for the plane beach by Eckhart (1951) and for the shelf by Snodgrass, Munk & Miller (1962). A fundamental difference between these shallow-water theories and the full linear theories is that the former allow the possibility of an infinite number of modes for any geometry, while the number of modes remains finite in the latter case.

For the rectangular shelf, Evans & McIver (1984) used the full linearized theory to show that for a given shelf width the number of possible modes increases as the depth of the shelf decreases. Also, for a fixed shelf geometry a single mode exists in the limits of both low and high frequency with a maximum number at some intermediate frequency. In the present work, the full linear solution given by Ursell (1951) for the submerged cylinder is computed without restriction upon the radius. Similar behaviour to the shelf geometry is found. For a fixed cylinder of radius a, it is found that a single mode exists for a depth of submergence of the cylinder greater than about $1.07a$. As the depth of submergence is decreased, further trapped modes appear. Again, for a fixed cylinder configuration the number of modes varies with frequency, these being just a single mode in the low- and high-frequency limits.

The full linear theory derived by Ursell (1951) for the cylinder is not easily computed. However, here, simple lower bounds on the dispersion relation for the fundamental mode are found using a simple comparison theorem for edge waves that is proved in §3. Also, an approximate dispersion relation for all of the modes is calculated using the so-called 'mild-slope' equation, based on the assumption that variations in the depth are small over a horizontal distance comparable to the wavelength. The equation has been used previously for computations of the plane sloping beach edge-wave modes of Ursell (1952). Thus, Smith & Sprinks (1975) showed that the dispersion relation may be determined very accurately using the mild-slope equation for all but the fundamental mode on the greatest beach slopes, while Kirby, Dalrymple & Liu (1981) used the equation to calculate the corresponding surface profiles, but did not make a comparison with the exact expressions given by Ursell (1952). To help validate the use of the mild-slope equation, such a comparison is made here and close agreement with the exact linear theory is found. For the trapped modes over the submerged cylinder, the mild-slope equation is again shown to compare favourably with the full linear theory when computing the dispersion relation. In addition, it was found that the surface profiles are reproduced reasonably well using the mild-slope equation with usually only small errors when compared with the full linear theory.

Trapped waves of the type discussed in this paper have been observed by the second author during experimental testing of a device for absorbing energy from waves based on the oscillations of a horizontal submerged cylinder. At certain frequencies the incident wavetrain would excite large-amplitude wave motions, confined to the immediate vicinity of the cylinder, which would persist after the wavemaker was switched off. No measurements were made of these motions.

2. Formulation

Cartesian axes are chosen so that y is directed vertically downwards and x and z are in the plane of the free surface. The depth contours are assumed to be parallel to the z-axis, so that the depth varies only in the x-direction. With the usual assumptions of the inviscid, linearized theory of water waves, the velocity potential $\Phi(x,y,z,t)$ satisfies

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \quad \text{within the fluid,} \tag{2.1}$$

the free surface condition

$$\frac{\partial^2\Phi}{\partial t^2}=g\frac{\partial\Phi}{\partial y}, \quad y=0, \tag{2.2}$$

and

$$\frac{\partial\Phi}{\partial n}=0, \tag{2.3}$$

on the solid boundaries, where n is measured normal to the boundary.

Solutions of (2.1–2.3) corresponding to waves of frequency σ and wavelength k travelling along the depth contours have the form

$$\Phi(x,y,z,t)=\phi(x,y)\cos(kz-\sigma t), \tag{2.4}$$

where ϕ satisfies

$$\nabla^2\phi\equiv\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=k^2\phi \tag{2.5}$$

within the fluid, and

$$\frac{\partial \phi}{\partial y} + K\phi = 0 \quad \text{on } y = 0, \tag{2.6}$$

where

$$K = \frac{\sigma^2}{g}. \tag{2.7}$$

The condition (2.3) is also to be satisfied by ϕ. For trapped-wave solutions the motion must decay at large distances, thus

$$\phi, |\nabla\phi| \to 0 \quad \text{as} \quad |x| \to \infty. \tag{2.8}$$

3. A comparison theorem for edge waves

THEOREM. *Let D_1 and D_2 be two semi-infinite fluid domains; the domain D_1 bounded by the free surface F and the curve C_1; the domain D_2 bounded by F and a second curve C_2 as in figure 1. Let the curves C_1 and C_2 cut F at the same point and extend to infinity in such a way that D_1 is contained entirely within D_2. Suppose edge-wave solutions exist for each domain as defined by (2.5–2.8). Let σ_i be the frequency of the fundamental mode with wavenumber k for the domain D_i ($i = 1, 2$). Then*

$$\sigma_1^2 \leqslant \sigma_2^2. \tag{3.1}$$

Proof: Suppose ϕ_i is the potential of the fundamental edge-wave mode for the domain D_i ($i = 1, 2$). Then by definition

$$\phi_i > 0 \quad \text{throughout } D_i,$$

and

$$\frac{\partial \phi_i}{\partial x}\Big/\phi_i \quad \text{is bounded as } x \to \infty \quad (i = 1, 2).$$

From theorem 2 of Grimshaw (1974)

$$\frac{\sigma_1^2}{g}\int_F \phi_2^2 \,\mathrm{d}x \leqslant \iint_{D_1} (|\nabla\phi_2|^2 + k^2\phi_2^2)\,\mathrm{d}x\,\mathrm{d}y \leqslant \iint_{D_2} (|\nabla\phi_2|^2 + k^2\phi_2^2)\,\mathrm{d}x\,\mathrm{d}y = \frac{\sigma_2^2}{g}\int_F \phi_2^2\,\mathrm{d}x, \tag{3.2}$$

where the last inequality follows since $D_1 \subset D_2$ and the integrand is positive throughout D_i ($i = 1, 2$), and the last step follows from considering the energy of the fluid motion (Grimshaw (1974), equation (2.12)). The result (3.1) follows immediately.

Similar results to (3.1) have previously been obtained for finite domains, see, for example, Courant & Hilbert (1953, chapter 6) where the results are obtained by the application of extremum principles.

This simple comparison theorem may be used to obtain bounds on a dispersion relation using known edge-wave solutions. Consider a fixed, submerged, horizontal, circular cylinder in deep water. Let the radius of the cylinder be a and the axis be submerged to a depth $f (> a)$, as shown in figure 2. The contour C_2 is chosen to be the positive y-axis, except for $|y - f| < a$, where it is the surface of the cylinder $y = f \pm (a^2 - x^2)^{\frac{1}{2}}$, $x > 0$.

Firstly, consider C_1 to be a plane beach at such an angle β to the horizontal as to touch, but not cut, the surface of the cylinder, as shown in figure 2(*a*). The frequency σ_b of the fundamental mode for edge waves on such a plane beach is given by

$$\sigma_b^2 = gk \sin\beta, \tag{3.3}$$

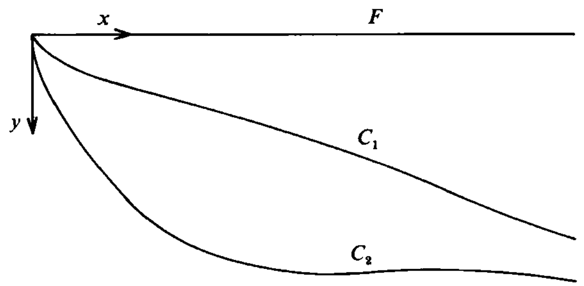

FIGURE 1. Schematic drawing of bounding contours for comparison theorem.

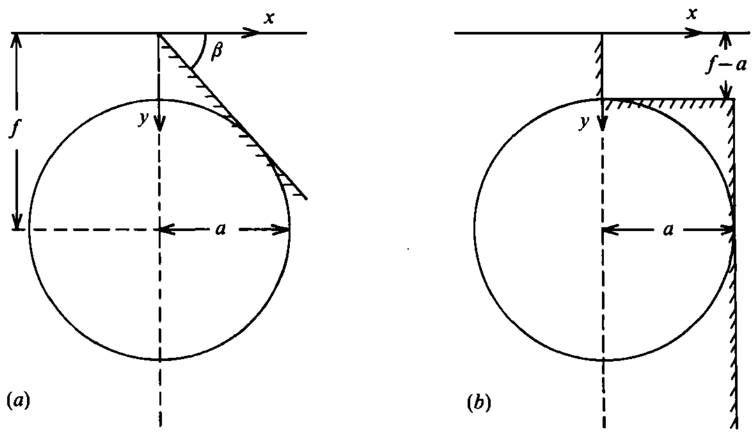

FIGURE 2. Definition sketches for application of comparison theory to the circular cylinder: (a) comparison with plane beach; (b) comparison with rectangular shelf.

so that, by (3.1), the fundamental mode of the same length trapped by the cylinder has a frequency σ that satisfies

$$\sigma^2 \geqslant gk \sin\beta = gk\left(1-\left(\frac{a}{f}\right)^2\right)^{\frac{1}{2}}. \tag{3.4}$$

A second bound may be found by considering C_1 to be a bounding rectangular shelf, as shown in figure 2(b). For arbitrary depths, there is no explicit dispersion relation known for edge waves over a rectangular shelf. However, Evans & McIver (1984), following the analysis of Jones (1953), show that, for a shelf of width a and depth h, the fundamental edge-wave mode of frequency σ_r and wavenumber k satisfies

$$k < k', \tag{3.5}$$

where k' is the solution of $$\sigma_r^2 = gk' \tanh k'h. \tag{3.6}$$

Hence from (3.5), (3.6) $$\sigma_r^2 > gk \tanh kh \tag{3.7}$$

and from (3.1) $$\sigma^2 > gk \tanh k(f-a). \tag{3.8}$$

Thus two lower bounds for the dispersion relation have been found. As will be seen later, (3.4) is the closer bound for small σ and (3.8) for large σ. No new useful upper bound for the frequency of the fundamental mode appears to be available from this theory. There is an upper bound on the frequency of trapped waves given by $\sigma^2 < gk$; this condition arises from the requirement that the motions are exponentially decaying at large distances.

4. Method of solution

Using full linear theory, Ursell (1951) examined the trapping of waves by a submerged, horizontal cylinder and showed that the dispersion relation may be determined by locating the zeros of a certain infinite determinant, given in the Appendix of the present paper. In general this is a non-trivial task, though Ursell was able to show that, for fixed kf, a trapping mode exists provided ka is sufficiently small. In particular, he was able to derive an explicit dispersion relation valid for small ka.

In the present work, the problem is solved numerically for arbitrary ka. A convenient recurrence relation for the evaluation of the matrix elements calculated by Ursell is given in the Appendix. The problem is completely specified by three non-dimensional parameters: kf, ka and the ratio of K to k. Ursell denotes the latter by $\cos\alpha$ because, for trapped waves, K is necessarily less than k. Two of the parameters may be fixed and the third varied until a zero of the determinant is found. The greatest computational expense incurred while evaluating the matrix elements is in computing the integrals $\beta_{mn}(kf, \alpha)$, as defined by (A 3) of the Appendix. Thus, it might seem appropriate to keep kf and α fixed while ka is varied. However, the range of possible values of α is so restricted (see §3) that it was more convenient in practice to have this as the variable and to take kf and ka as the fixed parameters. This is equivalent to fixing the configuration (a cylinder of given radius in a channel of given width), and searching for the frequencies of the edge-wave modes.

The solution procedure, then, was to fix kf and ka and systematically to vary α, looking for zeros of a suitably truncated determinant. Convergence checks were carried out as described for a similar problem by Evans & McIver (1984). The m, n element of the matrix is proportional to $(a/f)^{m+n}$ so that convergence was least good with the cylinder close to the surface. It turned out that Ka could be determined to three decimal places for $f/a = 1.1$ by taking a 10×10 system, while for $f/a = 1.01$ up to a 40×40 system was sometimes required. Generally, higher frequencies required more terms than lower frequencies.

An alternative method of solution is to make use of the 'mild slope' equation; derivations may be found in Berkoff (1972) and Smith & Sprinks (1975). The basic assumption in these derivations is that the bottom slope is small or that changes in depth are small over one wavelength. Despite this, the equation gives accurate results for the circular cylinder where this assumption is clearly violated. This is not particularly surprising as the depth varies most slowly in the shallowest region where the predominant edge-wave motion occurs. The great advantage of the mild-slope equation is that it is readily solved for both the dispersion relation and surface profiles. Calculation of the surface profile, in particular, is cumbersome using the full linear equations given by Ursell (1951).

For edge waves of longshore wavenumber k and frequency σ, the mild-slope equation for the wave height ζ reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)+(\sigma^2 q-k^2 p)\,\zeta = 0 \tag{4.1}$$

(see Smith & Sprinks (1975), §3), where

$$p = gh\frac{\tanh k'h}{k'h}q, \tag{4.2}$$

$$q = \frac{1}{2}\left(1+\frac{2k'h}{\sinh 2k'h}\right), \tag{4.3}$$

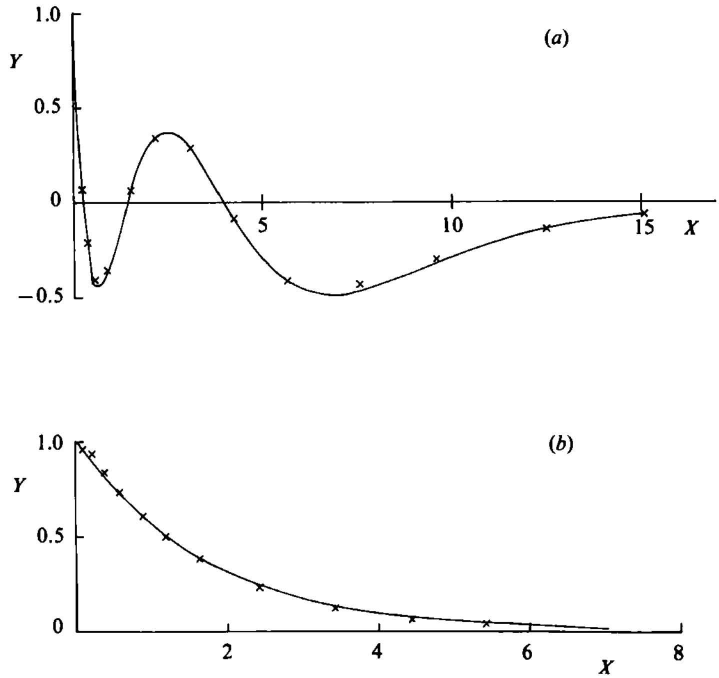

FIGURE 3. Comparison of Ursell's edge-wave modes for a plane beach of slope β (——) with the results obtained from the mild-slope equation ($\times$) $\beta = \frac{1}{18}\pi$ (fourth mode); (*b*) $\beta = \frac{1}{3}\pi$.

and k' is the real positive root of

$$\sigma^2 = gk' \tanh k'h. \tag{4.4}$$

Here $h = h(x)$ is the local water depth, so that k' is the local wavenumber and is a function of x through (4.4).

Equation (4.1) is of Sturm–Liouville type and standard library routines are available to solve for the eigenvalue k and for the corresponding eigenfunction. Smith & Sprinks (1975) calculated the dispersion relation for edge waves on a plane beach using (4.1). A comparison with the exact theory of Ursell (1952) gave graphically indistinguishable results for slopes of up to $\tan\beta \sim 1$. Kirby, Dalrymple & Liu (1981) calculated the surface profiles of plane beach edge waves but did not present a comparison with the exact formula derived by Ursell (1952). This comparison is made in figure 3 for beach angles of $\frac{1}{18}\pi$ and $\frac{1}{3}\pi$. For $\beta = \frac{1}{18}\pi$ there are four modes; the first three eigenfunctions calculated from the mild-slope equation are indistinguishable from the exact formula although some discrepancy does arise in the fourth mode as shown in figure 3(*a*). For the single mode at $\beta = \frac{1}{3}\pi$, σ^2/gk is determined to within 3% by the mild-slope equation and a comparison of the surface profiles (figure 3*b*) still shows good agreement. These results give confidence in the use of the mild-slope equation even when the bed slope is large.

For calculations of the trapping modes for the submerged cylinder using the mild-slope equation, it is convenient to substitute

$$x = aX, \quad \zeta = aY, \quad p = \sigma^2 a^2 r$$

into (4.1) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(r\frac{\mathrm{d}Y}{\mathrm{d}X}\right) + ((k'a)^2 - (ka)^2)\, rY = 0. \tag{4.5}$$

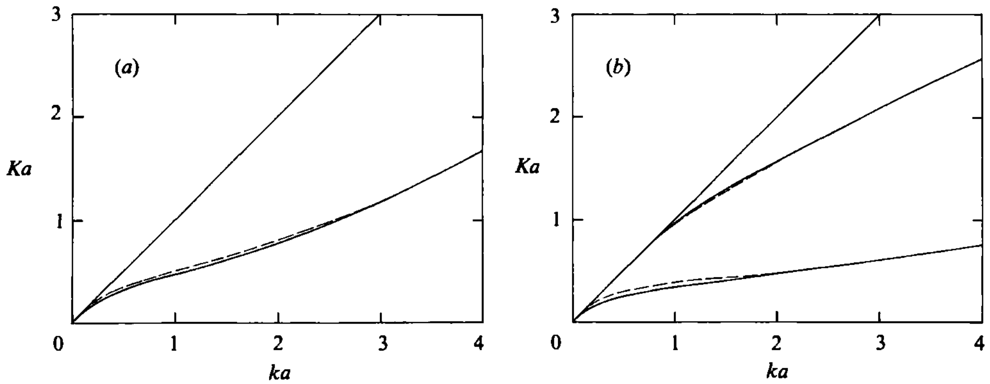

FIGURE 4. Comparison of the dispersion curves for the circular cylinder as computed by the full linear theory (——) and the mild-slope equation (– – –) for submergence: (*a*) $f/a = 1.05$; (*b*) $f/a = 1.01$.

Only symmetric trapping modes will be sought so that the boundary condition at $X = 0$ is just

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = 0. \tag{4.6}$$

A boundary condition at $X = 1$ may be derived by matching the solution with the exponentially decaying solution of (4.5) for deep water ($k' = K$). This results in

$$r\frac{\mathrm{d}Y}{\mathrm{d}X} = -\frac{1}{2(Ka)^2}(k^2-K^2)^{\frac{1}{2}}aY, \quad X = 1. \tag{4.7}$$

Alternatively, a semi-infinite condition may be imposed by taking Y to be zero at a suitably large value of X. The NAG library routines DØ2KDF and DØ2KEF respectively were used to solve for the eigenvalues and eigenfunctions. The eigenvalue routine DØ2KDF gives accurate results for both the finite and semi-infinite domains but is most efficient is used with the boundary condition appropriate to the finite domain. For the eigenfunction calculation using DØ2KEF the semi-infinite condition was used exclusively as in many cases there is a significant deviation from the zero level at the cylinder edge. Care must be taken to extend the domain sufficiently far from the cylinder for the surface to go to zero in an unconstrained way. If appropriate library routines are not available then the reader is referred to the method of Kirby *et al.* (1981).

A comparison of the results obtained for the dispersion relation from the exact theory and the mild-slope equation is made in figure 4. For $f/a \geqslant 1.1$ the results are indistinguishable; for values f/a close to unity a small discrepancy becomes apparent at low frequencies. For shorter waves the curves once more become indistinguishable until $ka \sim 10$. It was not possible to compute the exact theory beyond this because of overflow during the calculations. Hence, for $ka \geqslant 10$ calculations were made using the mild-slope equation only.

5. Results and discussion

By considering ka to be small, Ursell (1951) proved that a trapping mode may exist in the presence of a submerged horizontal cylinder. Jones (1953) proved the existence of trapped waves in a wide class of problems, including a cylinder of arbitrary radius.

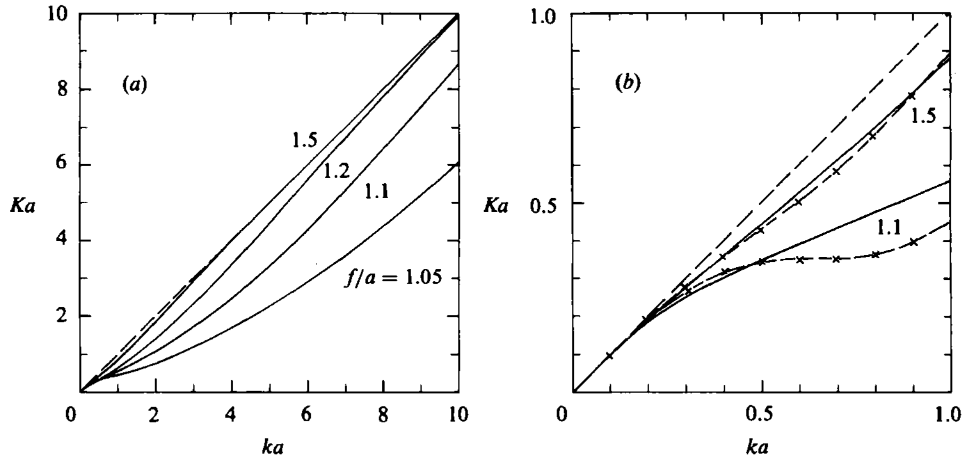

FIGURE 5. (*a*) The dispersion relation of the fundamental mode for $f/a = 1.05$, 1.1, 1.2 and 1.5. (*b*) Comparison between the full linear theory (——) and the approximate dispersion relation due to Ursell (–×–×–).

Computations based on the exact linear theory show that there is just a single mode for any wavelength whenever $f/a \geqslant 1.07$. As f/a is reduced below this value a second mode appears at certain wavelengths, followed by further modes as f/a approaches unity. It seems likely that the number of possible modes increases without bound as the top of the cylinder approaches the surface.

In figure 5(*a*) the dispersion relations of the fundamental mode are plotted for a range of values of the submergence f/a. The existence of trapped waves requires $K < k$ (Ursell (1951)) so that $K = k$ (the dashed line in figure 5*a*) is an upper bound for the dispersion relation. This line is also the dispersion relation for plane progressive waves in deep water travelling in the z-direction in the absence of the cylinder. It is apparent from figure 5(*a*) that the presence of a cylinder has little influence, in the sense of producing a significant deviation from the deep-water dispersion relation, unless it is placed quite close to the surface. The shapes of the dispersion curves bear a strong resemblance to those calculated by Evans & McIver (1984) for waves trapped by a rectangular shelf. A fundamental difference is in the behaviour for long waves. For a rectangular shelf of depth h, $K \sim k^2h$ for long waves and the depth of the shelf is important in determining the wave properties. For the cylinder, $K \sim k$ for long waves and so the influence of the geometry becomes negligible.

Ursell (1951, §5) derived an explicit dispersion relation for the fundamental mode for waves trapped by a cylinder by assuming that both ka and α were small. A comparison is made between this explicit relation and the exact theory in figure 5(*b*). For $f/a = 1.5$ the exact relation deviates little from the line $K = k$, where α is small, and so there is good agreement with Ursell's result. This is in contrast to $f/a = 1.1$ where there is a marked discrepancy as ka increases beyond about 0.5.

Two cases where there are higher modes are presented in figures 6 and 7. The dashed lines below the dispersion curves are the bounds derived in §3. For $f/a = 1.05$ a second mode exists for $2.3 \leqslant ka \leqslant 20$. That their must be a high-frequency cut-off for the higher modes is evident from the following argument. As ka becomes very large the fluid motion does not penetrate to the depth of the cylinder surface, so the waves are unaffected by the presence of the cylinder. There can therefore be no variation in the x-direction and the motion degenerates into a deep-water plane wavetrain

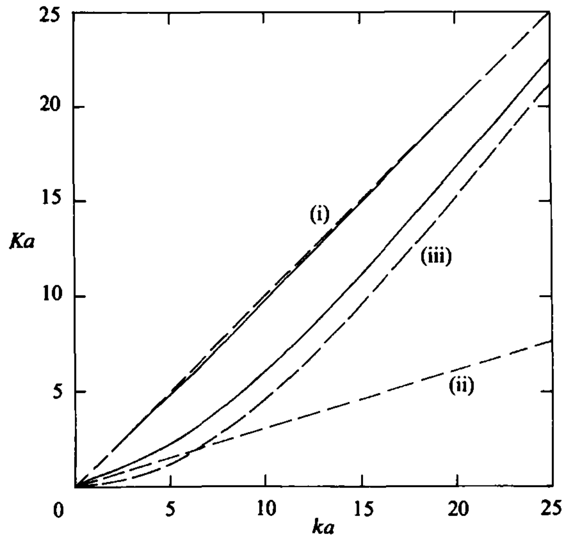

FIGURE 6. The dispersion relation for a cylinder submerged to a depth of $f/a = 1.05$; (——), dispersion curves; and (– – – – –), bounds derived in §3: (i) $K=k$; (ii) $K = k(1-(a/f)^2)^{\frac{1}{2}}$; (iii) $K = k \tanh k(f-a)$.

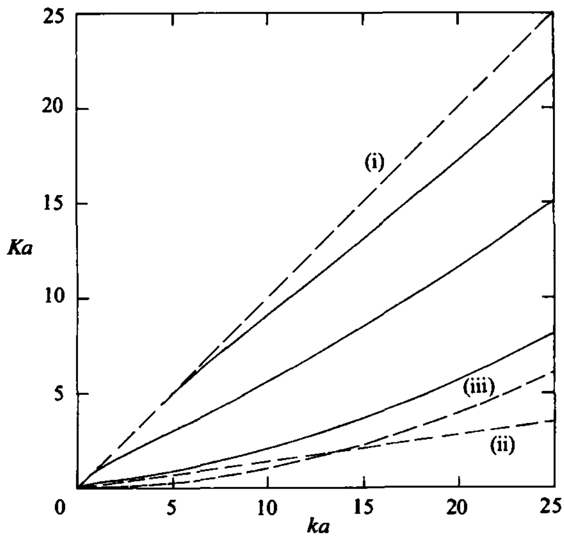

FIGURE 7. The dispersion relation for a cylinder submerged to a depth $f/a = 1.01$; (——), dispersion curves; and (– – – – –), bounds derived in §3: (i) $K = k$, (ii) $K = k(1-(a/f^2)^{\frac{1}{2}}$, (iii) $K = k \tanh k(f-a)$.

progressing in the direction of the cylinder axis, for which there is a unique dispersion relation. For $f/a = 1.01$ there are at least four modes, the fourth mode appearing at $ka \sim 35$. Comparison with the results for $f/a = 1.05$ in figure 5 shows a marked 'fanning out' of the curves reflecting the increased influence of the presence of the cylinder on the higher-frequency waves. Also, the higher modes occur at lower frequencies for the smaller depth of submergence.

Two sample sets of surface profiles are presented in figure 8 and 9. The number of crossings of $Y = 0$ may be used to identify each mode. The fundamental mode has

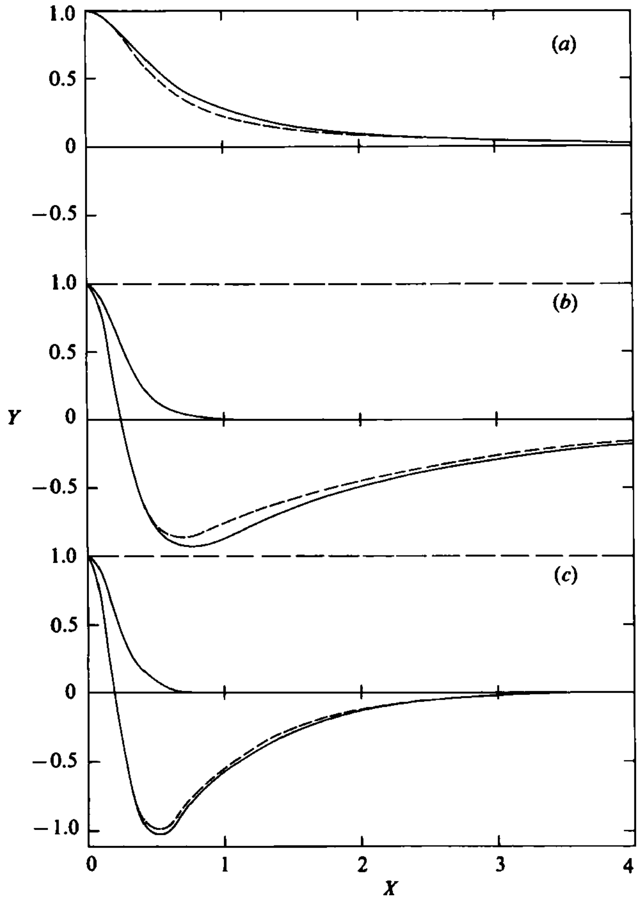

FIGURE 8. The surface profiles of the trapping modes for a cylinder submerged to a depth $f/a = 1.05$ at frequencies given by (a) $Ka = 1.05$; (b) $Ka = 3$; (c) $Ka = 5$: (——), full linear theory; and (- - - - -), the mild-slope equation.

no zero crossings, the second mode one zero crossing, and so on. The profiles were calculated using both the full linear theory given by Ursell (1951) and the mild-slope equation. In figure 8 ($f/a = 1.05$) the results for the fundamental mode at the frequencies corresponding to $Ka = 3$ and 5 are graphically indistinguishable using the two methods. Some of the profiles in figure 9 ($f/a = 1.01$) are calculated from the mild-slope equation only, as it was not possible to take sufficient terms in the full linear theory to obtain satisfactory convergence. The convergence of the full theory is very poor as f/a approaches unity. However, the agreement between the full theory and the mild-slope equation improves with increasing frequency in all the cases calculated where comparisons were possible, as in figure 8. Hence the profiles mentioned above, shown in figure 9, are thought to be accurate.

There are two likely sources of error arising from the use of the mild-slope equation in the present problem. First, no account is taken of any motion directly beneath the cylinder and, secondly, the edges of the cylinder clearly violate the mild-slope criterion. The importance of these sources of error will depend upon the frequency of the motion and the submergence of the cylinder, both effects being reduced when the submergence is increased. Consider a fixed submergence. At low frequencies when

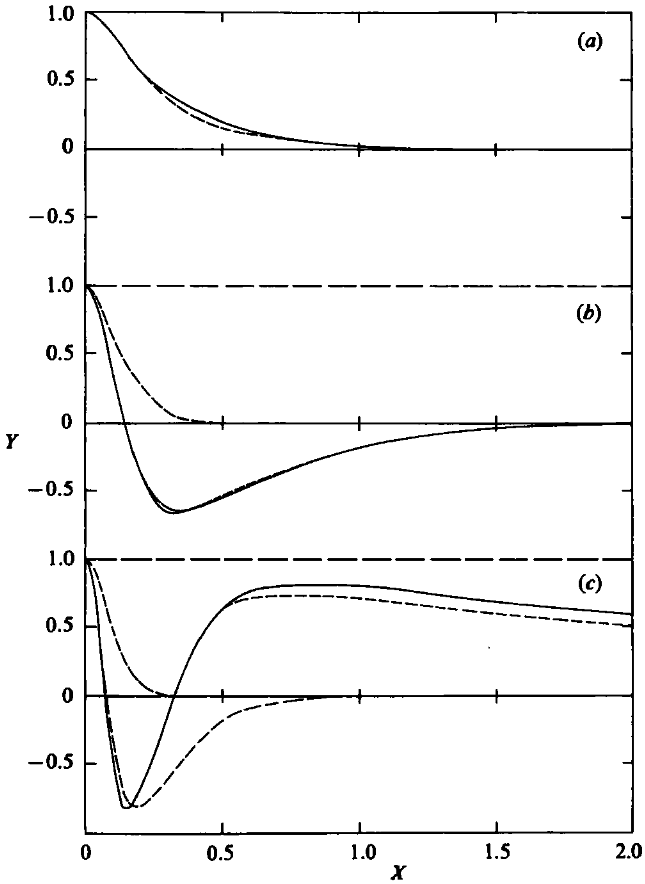

FIGURE 9. The surface profiles of the trapping modes for a cylinder submerged to a depth $f/a = 1.01$ at frequencies given by (a) $Ka = 0.5$, (b) $Ka = 2$, (c) $Ka = 5$. (——), full linear theory; and (- - - -), the mild-slope equation.

the motion penetrates to greater depths both effects may be significant. Thus there are discrepancies in the calculations for the fundamental mode in (a) of each of figures 8 and 9 and also in the dispersion relations presented in figure 4. At higher frequencies, the large slope at the edge of the cylinder may give rise to errors where a significant motion extends over the cylinder edge. Even in the worst cases the mild-slope equation reproduces the main features of the profile well; for most parameters the motion is chiefly confined to the vicinity of the cylinder and so agreement is very good. It is only where a mode has just appeared (as Ka increases) that these are significant motions at large distances from the cylinder. For example, in figure 9, $Ka = 5$, the third mode has only just appeared as can be seen from the dispersion relation in figure 7. As Ka increases the area between each mode profile and the zero level is reduced. Note that the errors in the dispersion relations calculated by the two methods do not account completely for the difference in the profiles. Corresponding changes in the parameters produce relatively small changes in the mode shapes.

For a fixed frequency, increasing the depth of submergence of the cylinder spreads the mode shape. For instance, compare the fundamental mode profiles in (a) of each

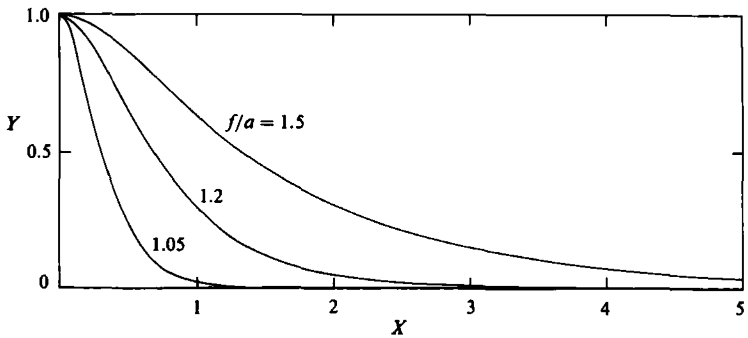

FIGURE 10. The surface profiles of the trapping modes for cylinder submergences $f/a = 1.05, 1.2, 1.5$ at a frequency given by $Ka = 2$.

of figures 8 and 9, or the second mode profiles in (c) of those figures. As the submergence increases beyond $f/a \sim 1.07$ only the fundamental mode exists. Figure 2 shows the effect of increasing submergence on the surface profile of the fundamental mode when the frequency is held fixed. In the limiting case of infinite submergence the cylinder does not affect the motion and there is a plane wavetrain travelling in the direction of the axis of the cylinder.

Appendix

Ursell (1951) shows that the trapping modes for the submerged horizontal cylinder correspond to zeros of the infinite determinant

$$\Delta \equiv |\delta_{nm} + \psi_{nm}(ka, kf, \alpha)|, \tag{A 1}$$

where

$$\psi_{nm} = \frac{I'_m(ka)}{K'_n(ka)} \beta_{nm}(kf, \alpha), \quad m \geqslant 1, n \geqslant 1,$$

$$\psi_{no} = \frac{I'_0(ka)}{ka\, K'_n(ka)} \beta_{no}(kf, \alpha), \quad n \geqslant 1,$$

$$\psi_{om} = \frac{ka\, I'_m(ka)}{K'_0(ka)} \beta_{om}(kf, \alpha), \quad m \geqslant 1,$$

$$\psi_{00} = \frac{I'_0(ka)}{K'_0(ka)} \beta_{00}(kf, \alpha).$$

Here, I_n and K_n are modified Bessel functions, α is defined by

$$\sigma^2 = gk \cos\alpha, \tag{A 2}$$

and

$$\beta_{nm} = \epsilon_m(-1)^{m+n} \int_0^\infty \cosh n\mu \cosh m\mu \exp(-2kf\cosh\mu) \times \left(\frac{\cosh\mu + \cos\alpha}{\cosh\mu - \cos\alpha}\right) \mathrm{d}\mu, \tag{A 3}$$

where

$$\epsilon_0 = 1, \quad \epsilon_m = 2 \quad (m \geqslant 1).$$

Now, write

$$\beta_{nm} = \tfrac{1}{2}\epsilon_m(-1)^{m+n}(J_{m+n} + J_{m-n}),$$

where

$$J_p = \int_0^\infty \cosh p\mu \exp(-2kf\cosh\mu)\left(\frac{\cosh\mu + \cos\alpha}{\cosh\mu - \cos\alpha}\right) \mathrm{d}\mu.$$

By combining J_{p+1} and J_{p-1} it follows that

$$J_{p+1} = 2\cos\alpha(K_p + J_p) - J_{p-1} + K_{p+1} + K_{p-1}, \tag{A 4}$$

where K_n is again the modified Bessel function.

Now, for large $p, J_p \sim K_p$ and

$$K_p(z) \sim \sqrt{\frac{\pi}{2p}}\left(\frac{2p}{ze}\right)^p.$$

Therefore it is numerically convenient to define a scaled integral

$$J'_p = J_p \bigg/ \left(\frac{2p}{ze}\right)^p \tag{A 5}$$

and use the corresponding modification of (A 4) to calculate the integrals. The recurrence is begun by numerical integration. Similar scalings were employed when calculating the modified Bessel functions in (A 1).

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